

# Topological defect solutions in new families of sine-Gordon models

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We study specific deformation of the sine-Gordon model, to get to families of models which starts with the sine-Gordon model itself, including the double sine-Gordon and the triple sine-Gordon model, and so on. The deformation is controlled by two parameters, one very small, used to control a linear expansion on it, and the other, which specifies the particular model in the family are constructed explicitly from the topological defects of the sine-Gordon model. The procedure can be used iteratively, leading to a diversity of possibilities to construct families of models of the sine-Gordon type.

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## I. INTRODUCTION

Topological defects appear in nature in a diversity of contexts. They are of current interest in several branches of nonlinear science, in particular in high energy and condensed matter physics, where they can be used to describe phase transitions in the early universe, map interfaces separating distinct regions and contribute to pattern formation in nature [1–4]. An interesting recent example of the use of topological defect concerns the study of magnetic domain wall in a nanowire, intended for the development of magnetic memory[5].

In this work we study the presence of topological defects in relativistic models described by a real scalar field  $\phi$  in  $(1, 1)$  spacetime dimensions. We focus attention on the deformation procedure, a method introduced in [6] and further used to describe the presence of topological defects in models described by scalar fields in a diversity of scenarios; see, e. g., Ref [7].

In particular, in a recent Letter [8] one has shown how to deform the  $\phi^4$  model with spontaneous symmetry breaking, to generate new family of sine-Gordon models [9]. There it was explicitly shown that all the topological defects are constructed from the topological defects present by the  $\phi^4$  model. Another point is that the family of models is obtained with a simple deformation function, which depends on two real parameters, one controlling the position of the minima and the height of the maxima, and the other specifying the particular member in the family of models.

The work [8] has inspired us to further study the issue, and this work offer another possibility of constructing new families of sine-Gordon models. Here, however, we start with the sine-Gordon model itself [9], and we use a deformation function with is controlled by two parameters, one very small, which induces small deviation from identity, and another once, which allows specifying the member in the family of models. This new procedure has advantage of being very simple to be implemented, leading to potentials of direct interest to practical applications. Another issue is that the small parameter is in

general of great help to find solutions of the deformed model explicitly.

The deformation procedure has also been used by other authors, to investigate issues of current interest. For instance, the studies presented in [10–12] show that it works appropriately for a variety of contexts in high energy physics. On the other hand, one knows that the sine-Gordon model has been used to investigate issues related to DNA; see Refs [13, 14]. In this sense, the family of models here introduced may be of interest to describe the presence of solitons in DNA; see, e. g., [15].

Let us now consider models described by the Lagrange

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \quad (1)$$

where  $V(\phi)$  is the potential which specifies the model under consideration. Also,  $x^\mu = (x^0 = t, x^1 = x)$  and we consider  $t, x$  and the field  $\phi$  dimensionless, for simplicity.

We first review some basic facts about topological solutions, focusing our attention on static fields. The equation of motion for  $\phi = \phi(x)$  is given by

$$\frac{d^2 \phi}{dx^2} = \frac{dV}{d\phi} \quad (2)$$

Topological solutions usually appear when the potential is non-negative, which can be written in form, using  $W = W(\phi)$ ,

$$V(\phi) = \frac{1}{2} W_\phi^2 \quad (3)$$

where  $W_\phi$  stands for  $dW/d\phi$ . In this case we get

$$\frac{d^2 \phi}{dx^2} = W_\phi W_{\phi\phi} \quad (4)$$

The point here is that this equation can be solved by solutions of the first order equation

$$\frac{d\phi}{dx} = W_\phi \quad (5)$$

Since the potential does not see the sign of  $W$ , Eq (5) can also be seen with  $W$  changed to  $-W$ . The approach then

maps the second order equation of motion into two first order equations, which is in general valid for topological solution.

A topological solution which solves the first order equation on the topological section  $(jk)$  has energy minimized to the value  $E_{BPS}^k = |W(\bar{\phi}_j) - W(\bar{\phi}_k)|$ , where  $\bar{\phi}_j$  and  $\bar{\phi}_k$  are two adjacent minima in the set  $\{\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_n\}$  of minima of the model. This is the Bogomol'nyi bound, and solution of the first order equation is named BPS state [16], and is linearly stable. Since  $W(\phi)$  is a function of  $\phi$ , it can be used to define the topological behavior, which can be seen from the topological current

$$j_T^\mu = \varepsilon^{\mu\nu} \partial_\nu W(\phi) \quad (6)$$

This definition for the topological current is different from the standard form, which uses  $j^\mu = \varepsilon^{\mu\nu} \partial_\nu \phi$ ; see, e. g., [1]. However, one can use (6) to show that the topological charge of the solution equals its energy, apart from a sign factor, and is more appropriate in general [17].

## II. DEFORMATION PROCEDURE

We now focus attention on the deformation procedure proposed in [6]. An important advantage of using this approach is that we can easily get to new models and construct their topological defects analytically. Here we briefly review the procedure, starting with the standard model

$$\mathcal{L} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\chi) \quad (7)$$

where  $\chi = \chi(x, t)$  is a real scalar field, and  $V(\chi)$  is the potential which describes the model. For simplicity, we are using natural units, and we have rescaled the field, and the space and time coordinates to make them dimensionless. We suppose that the model supports topological defects and that we know the corresponding solutions analytically. In this paper, we investigate small modifications to the sine-Gordon model whose potential is given by

$$V(\chi) = \frac{1}{2} \cos^2(\chi) \quad (8)$$

This potential is obtained from the simple function  $W(\chi) = \sin(\chi)$  and have infinite number of minima  $\bar{\chi} = \pm n\pi/2$ , where  $n = 1, 3, 5, \dots$ . The topological solutions that connects these minima are

$$\chi_s(x) = \pm(\theta(x) + k\pi), \quad \theta(x) = \arcsin(\tanh(x)). \quad (9)$$

where  $k = 0, \pm 1, \pm 2, \dots$ . The energy is  $E = 2$ , since we are using dimensionless units.

According to the deformation procedure, we can consider another model, described by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \quad (10)$$

where  $U(\phi)$  is the new potential, which is written in terms of the starting potential  $V(\chi)$  as

$$U(\phi) = \frac{V(\chi \rightarrow f(\phi))}{f'^2(\phi)} \quad (11)$$

where  $f(\phi)$  is the deformation function. In this case, if  $\chi(x)$  is a static solution of the starting model, then we get  $\phi(x)$  is a solution of the new, deformed model, and it is given by

$$\phi(x) = f^{-1}[\chi(x)] \quad (12)$$

This is the general procedure, and now we specialize to the case where the deformation function describes small deviation from the identity.

Let us choose  $f(\phi) = \phi + \epsilon g(\phi)$ , where  $\epsilon$  is very small parameter to be used to allow power expansion on it. However, we should do it with care, since the function  $g(\phi)$  cannot increase too much to break the approximation up to the first order power on  $\epsilon$  which we will be implementing from now on.

Although the deformation procedure is valid for general potential, let us concentrate on models controlled by  $W$ , described by

$$V(\chi) = \frac{1}{2} W_\chi^2 \quad (13)$$

In this case we get that

$$U(\phi) = \frac{1}{2} \mathcal{W}_\phi^2 \quad (14)$$

where the new  $\mathcal{W}$  has the form

$$\mathcal{W}_\phi = W_\phi - \epsilon (W_\phi g_\phi - W_{\phi\phi} g) \quad (15)$$

This is general result, and we can obtain the minima of the new potential from the equation

$$W_\phi = \epsilon (W_\phi g_\phi - W_{\phi\phi} g) \quad (16)$$

The minima can be used to find the corresponding energy in each topological sector, in the form  $E_{ij} = |\mathcal{W}_i - \mathcal{W}_j|$ , where  $\mathcal{W}_i$  and  $\mathcal{W}_j$  stand for  $\mathcal{W}(\phi)_i$  and  $\mathcal{W}(\phi)_j$ , respectively, and  $\phi_i$  and  $\phi_j$  represent two adjacent minima of the new potential. Also, from the expression which defines the deformation function we can write, up to first order in  $\epsilon$ , the solution of the new model as  $\phi = \chi - \epsilon g(\chi)$ , where  $\chi$  represents solution of the initial model.

## III. NEW FAMILY OF MODELS

To introduce the new families of models, let us consider the starting model as the sine-Gordon given by Eq. (8). As one knows, the sine-Gordon is an important model [9], and the above procedure will deform the model, leading to other models, close to the original model. Moreover, since we are starting from a periodic potential, we can

naturally choose  $g(\phi)$  periodic potential, making it limited in a way such that the first order approximation in  $\epsilon$  remains valid in the real line. In this way, our main goal is to introduce new family of models, as double and triple sine-Gordon models.

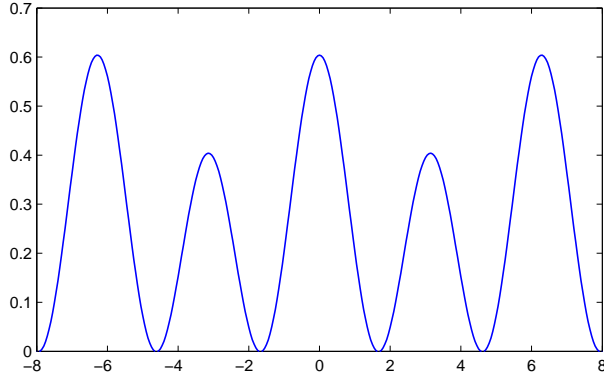


Figure 1: Potential of double sine-Gordon model, for  $s = 1$  and  $\epsilon = -0.1$

Here we take the deformation function in the form

$$f_s(\phi) = \phi + \epsilon \sin\left(\frac{\phi}{s}\right), \quad (17)$$

parametrized by real  $s$  and very small  $\epsilon \approx 0$ . Using the sine-Gordon model as the starting model, this function leads to the model described by the potential (14), where

$$\mathcal{W}_\phi = \cos(\phi) - \epsilon \left[ \sin(\phi) \sin\left(\frac{\phi}{s}\right) + \frac{1}{s} \cos(\phi) \cos\left(\frac{\phi}{s}\right) \right]. \quad (18)$$

The additional term that arises shifts the minima of the potential to

$$\phi_{\pm}^{min} = \pm \left( \frac{n\pi}{2} - \epsilon \sin\left(\frac{n\pi}{2s}\right) \right), \quad n = 1, 3, 5, \dots \quad (19)$$

It also changes the maxima

$$\phi_{\pm}^{max} = \pm \left( m\pi + \frac{\epsilon(s^2 - 1)}{s^2} \sin\left(\frac{m\pi}{s}\right) \right), \quad m = 0, 1, 2, \dots \quad (20)$$

and their respective heights

$$h_{m+1} = \frac{1}{2} - \frac{\epsilon}{s} \cos\left(\frac{m_s\pi}{s}\right), \quad m_s = 0, 1, 2, \dots, s. \quad (21)$$

We note that the modified potential has periodicity  $2\pi s$ .

For  $s$  integer, we see that the number of distinct topological sectors, which are labeled by  $m$ , is given by  $s + 1$ . Then, for  $s = 1$  we get double sine-Gordon, for  $s = 2$  triple sine-Gordon model, and son on. The family of models which is generated in the present work is different from the family introduced in Ref [? ], since there we started with another model, and used different deformation function. To see how the models behave in present

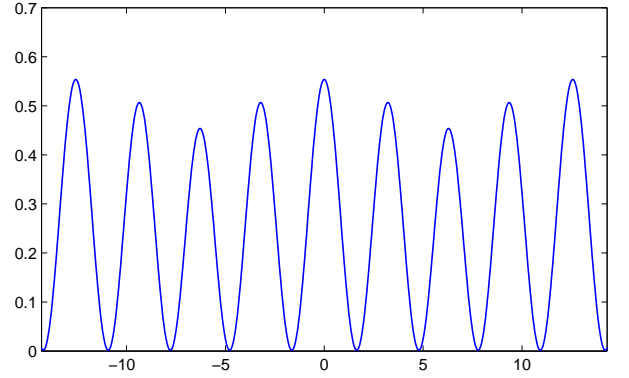


Figure 2: Potential of triple sine-Gordon model, for  $s = 2$  and  $\epsilon = -0.1$

case, in Figs 1 and 2 we depict the potentials for  $s = 1$  and  $s = 2$ , respectively.

The new function  $\mathcal{W}$  can be written as

$$\mathcal{W} = \sin(\phi) - \frac{\epsilon}{2} \left[ \frac{(s+1)}{(s-1)} \sin\left(\frac{(s-1)}{s}\phi\right) - \frac{(s-1)}{(s+1)} \sin\left(\frac{(s+1)}{s}\phi\right) \right] \quad (22)$$

This  $\mathcal{W}$  and the minima, given by Eq. (19), can be used to calculate the energy of all energy of all the topological solutions straightforwardly. The useful expression is

$$\mathcal{W}(\phi_{\pm}^{min}) = \pm(-1)^{\frac{n-1}{2}} \left( 1 - \frac{2s}{s^2 - 1} \cos\left(\frac{n\pi}{2s}\right) \epsilon \right) \quad (23)$$

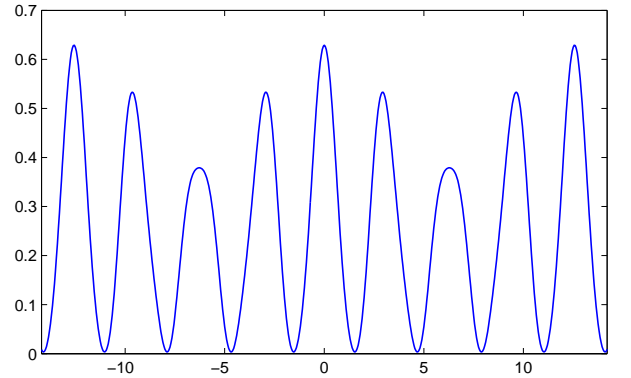


Figure 3: Potential of special sine-Gordon model, for  $s = 2/5$  and  $\epsilon = -0.05$

Moreover, the topological solutions of first order equation (5) can be obtained from the inverse of the deformation

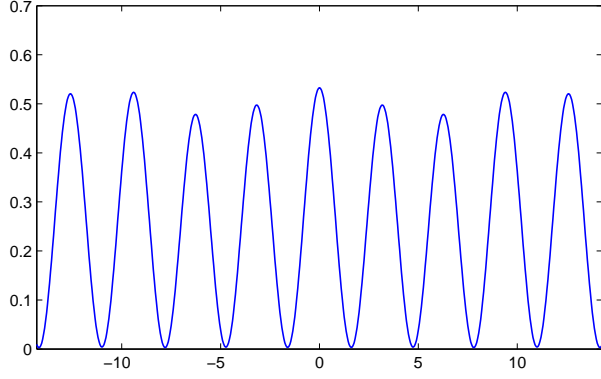


Figure 4: Potential of double sine-Gordon model, for  $s = \sqrt{3}$  and  $\epsilon = -0.05$

tion function. We use Eq. (17) to get

$$\phi(x) = \phi_0(x) - \epsilon \sin\left(\frac{1}{s}\phi_0(x)\right) \quad (24)$$

where  $\phi_0(x)$  represents the solutions of the sine-Gordon model, given by Eq. (9); recall that  $\phi_0(x) = \chi_s(x)$ . With this, it is not hard to calculate the corresponding energy density, which is given by

$$\rho(x) = \rho_0(x) \left[ 1 - \frac{2\epsilon}{s} \cos\left(\frac{\phi_0(x)}{s}\right) \right], \quad (25)$$

where  $\rho_0(x) = \text{sech}^2(x)$  is the energy density of the solutions of the sine-Gordon model.

Let us now consider the case of the triple sine-Gordon model, which is obtained with  $s = 2$ . In this case, there are three kinds of topological sectors. The sectors of the first kind are described by following solutions

$$\phi_1(x) = \theta(x) + 4l\pi - \epsilon \text{sgn}(x) \sqrt{\frac{1}{2}(1 - \text{sech}(x))}, \quad (26)$$

where  $\text{sgn}(x)$  is the sign function and  $l$  is a integer that connect the minima

$$-\frac{\pi}{2} + 4l\pi + \epsilon \frac{\sqrt{2}}{2}, \quad \text{and} \quad \frac{\pi}{2} + 4l\pi - \epsilon \frac{\sqrt{2}}{2}. \quad (27)$$

The sectors of the second kind are described by solutions

$$\phi_3(x) = \theta(x) + (4l+1)\pi - \epsilon \sqrt{\frac{1}{2}(1 + \text{sech}(x))} \quad (28)$$

which connect the minima

$$\frac{\pi}{2} + 4l\pi - \epsilon \frac{\sqrt{2}}{2}, \quad \text{and} \quad \frac{3\pi}{2} + 4l\pi - \epsilon \frac{\sqrt{2}}{2}. \quad (29)$$

The sectors of the third kind are described by solutions

$$\phi_3(x) = \theta(x) + 2l\pi + \epsilon \text{sgn}(x) \sqrt{\frac{1}{2}(1 - \text{sech}(x))} \quad (30)$$

which connect the minima

$$\frac{3\pi}{2} + 4l\pi - \epsilon \frac{\sqrt{2}}{2}, \quad \text{and} \quad \frac{5\pi}{2} + 4l\pi + \epsilon \frac{\sqrt{2}}{2}. \quad (31)$$

The energy densities of the solutions corresponding to the three distinct topological sectors are given by

$$\rho_1(x) = \left( 1 - \epsilon \sqrt{\frac{1}{2}(1 + \text{sech}(x))} \right) \text{sech}^2(x), \quad (32a)$$

$$\rho_2(x) = \left( 1 + \epsilon \text{sgn}(x) \sqrt{\frac{1}{2}(1 - \text{sech}(x))} \right) \text{sech}^2(x), \quad (32b)$$

$$\rho_3(x) = \left( 1 + \epsilon \sqrt{\frac{1}{2}(1 + \text{sech}(x))} \right) \text{sech}^2(x), \quad (32c)$$

We use Eq. (23) to obtain the corresponding energies

$$E_1 = 2 - \frac{4\sqrt{2}}{3}\epsilon, \quad E_2 = 2, \quad E_3 = 2 + \frac{4\sqrt{2}}{3}\epsilon. \quad (33)$$

There are many other value of  $s$  which introduce new features to the deformed models. The specific sequence  $s = 2/3, 1/2, 2/5, 1/3, \dots$ , leads to interesting potentials, and to other kinds of sine-Gordon models. To illustrate this fact, in Fig. 3 we depict the potential for  $s = 2/5$ . The topological solutions and their aptitudes, widths, energy densities and energies can all be calculated straightforwardly, using the approach shown above. We can also take  $s$  irrational, leading to quasiperiodic potentials. An illustration of this is given in Fig. 4, for  $s = \sqrt{3}$ . As before, the topological solutions, amplitudes, widths, energy densities and energies can also be calculated straightforwardly.

The quasiperiodic behavior of the potential can be better seen if one changes the deformation function appropriately. To see this we note that the same deformation function can be used to deform the deform model once again. Since  $\epsilon$  is small parameter, this will end up with the deformation

$$f(\phi) = \phi + \epsilon(g_1(s_1) + g_2(s_2)) \quad (34)$$

Note that the procedure can be repeated iteratively, giving rise to interesting deformations. Here, however, we only discuss the second iteration, which is explicitly given by Eq. (34). If we deform the sine-Gordon potential with the function (34), we get to a diversity of interesting potentials. A typical example of this depicted in Fig. 5, for  $s_1 = 1$  and  $s_2 = 2$ . Another interesting case is given by  $s_1 = \sqrt{2}$  and  $s_2 = \sqrt{3}$ , which is depicted in Fig. 6. This last case enhances the quasiperiodic behavior of the potential in significant way, as we have commented above. As before, we can get all solutions, and the corresponding amplitudes, widths, energy densities and energies very easily.

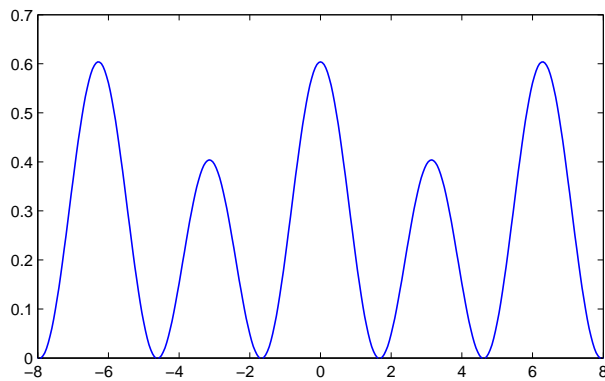


Figure 5: Potential of special sine-Gordon model, for  $s_1 = 1$ ,  $s_2 = 2$  and  $\epsilon = -0.1$

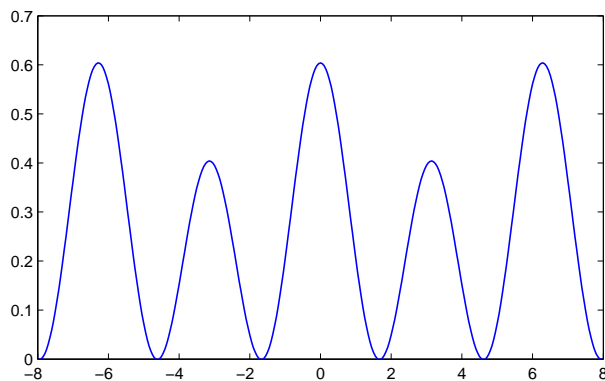


Figure 6: Potential of double sine-Gordon model, for  $s = \sqrt{2}$ ,  $s = \sqrt{3}$  and  $\epsilon = -0.1$

#### IV. FINAL COMMENTS

In summary, in this Letter we studied interesting models of sine-Gordon, searching for the topological solutions they may comprise. The procedure engender a diversity of possibilities, with all topological defects being explicitly constructed from the topological defects of the sine-Gordon model. The solutions, and their amplitudes, widths, energy densities and energies are simple to be obtained. The family of models can be studied easily, and the suggested models can be used for practical applications in a diversity of scenarios, in particular within the braneworld context where the brane is stabilized by a real scalar field [18], and in Biology, to study conformational structures in biomolecules at the nano metric scale. In addition, since the procedure involves small deformation of a given model, it can be used in more general sense, to study small deformations of other models, of interest to high energy and other branches of physics.

Another interesting issue concerns the use of the double sine-Gordon model, instead of the sine-Gordon, as original model, to generate another family of models. Also, we could apply the same methodology to more general models, including the case where one modifies the dynamics of the scalar field, to deal with higher order power in the first derivative of the field, which is also of current interest to high energy physics. Some of these issues will be discussed in future work.

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- [1] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
  - [2] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Others Topological Defects* (Cambridge, UK, 1994).
  - [3] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge UP, Cambridge, UK, 2004).
  - [4] D. Walgraef, *Spatio-temporal Pattern Formation* (Springer-Verlag, New York, 1997).
  - [5] A. Vanhaverbeke, A. Bischof and R. Allenspach, Phys. Rev. Lett. **101**, 107202 (2008).
  - [6] D. Bazeia, L. Losano, and J. M. C. Malbouisson, Phys. Rev. D **66**, 101701 (2002).
  - [7] C. A. Almeida, D. Bazeia, L. Losano, and J. M. C. Malbouisson, Phys. Rev. D **69**, 067702 (2004); V. I. Afonso, D. Bazeia, M. A. Gonzalez Leon, L. Losano, and J. Mateos Guilarte, Nucl. Phys. B **810**, 427 (2009); D. Bazeia, Ashok Das, L. Losano, and M. J. Santos, Applied Math. Lett. **23**, 681 (2010).
  - [8] D. Bazeia, L. Losano, R. Menezes, and M. M. Sousa, EPL **87**, 21001 (2009).
  - [9] G. Mussardo, V. Riva, and G. Sotkov, Nucl. Phys. B **87**, 548 (2005); G. Mussardo, Nucl. Phys. B **779**, 101 (2007); A. Kundu, Phys. Rev. Lett. **99**, 154101 (2007); L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. Lett. **99**, 207002 (2007); L. A. Ferreira, B. Piette, and W. J. Zakrzewski, Phys. Rev. E **77**, 036613 (2007); M. Cadoni, Y. -X. Liu, L. -D. Zhang, L. -D. Zhang, and Y. -S. Duan, Phys. Rev. D **78**, 0650025 (2008); D. Bazeia, L. Losano, J. M. C. Malbouisson, and R. Menezes, Physica D **237**, 937 (2008); J. H. Al-Alawi and W. J. Zakrzewski, J. Phys. A **41**, 315206 (2008).
  - [10] A. de Souza Dutra, arXiv:0705.3237; A. de Souza Dutra, Physica D **238**, 798 (2009).
  - [11] W. T. Cruz, M. O. Tahim, and C. A. S. Almeida, Europhys. Lett. **88**, 41001 (2009).
  - [12] A. E. R. Chumbes and M. B. Hott, Phys. Rev. D **81**, 045008 (2010).
  - [13] M. Peryard and A. R. Bishop, Phys. Rev. Lett. **62**, 2755 (1989); M. Peryard, Nonlinearity **17**, R1 (2004).
  - [14] S. W. Englander, N. R. Hippel, A. J. Heeger, J. A. Krumhauser, and A. Litwin, Proc. Natl. Acad. Sci. USA, **77**, 7222 (1980).
  - [15] M. Daniel and V. Vasumathi, Physica D **231**, 10 (2007).
  - [16] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, (1975) 760.

- [17] D. Bazeia, Phys. Rev. D **60**, 067705 (1999).
- [18] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999); W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999); O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).

